VII.—On Bicircular Quartics. By John Casey, A. B. [Abstract.]

[Read February 10, 1867.]

If we take the most general equation of the second degree in α, β, γ, where these variables denote circles in place of lines

$$(a, b, c, f, g, h,)\ (\alpha, \beta, \gamma,)^2 = 0,$$

we get the most general form in which the equation of a bicircular quartic can be written.

Setting out with this equation, I have proved that a bicircular quartic is the envelope of a variable circle which cuts the Jacobian (J) of α, β, γ, orthogonally, and whose centre moves on a given conic F; the equation of the conic F in three point co-ordinates being exactly the same in form as the equation of the quartic, the α, β, γ of the quartic being replaced by λ, μ, ν of the conic, where λ, μ, ν are the perpendiculars from given points on any variable tangent to the conic.

I have further proved that the same quartic may be described in more ways than one, in this manner, according to its class. Thus, if the quartic be of the eighth class, there are four conics, F, F', F'', F''', and corresponding to them four circles, J, J', J'', J'''; and the same quartic may be described indifferently as the envelope of a variable circle whose centre moves along any of these conics, which cuts the corresponding circle orthogonally.

I have proved that each of the four circles, J, J', J'', J''', inverts the quartic into itself.

If the quartic be of the sixth class, there are but three director conics, F, F', F''; and three circles of inversion, J, J', J''. In this case I have proved that the quartic must be the inverse of an ellipse or hyperbola, being the one or the other according as the double point it must have in addition to the circular points at infinity is a conjugate point, or a real double point.

If the quartic be of the fifth class, I have proved that it must be the inverse of a parabola; that it has but two director conics, F, F', and two circles of inversion.

For the quartics of each class, I have proved that the conics, F, F', &c., are confocal, their common foci being the double foci of the quartic; and that their points of intersection with their respective corresponding circles, J, J', &c., are the single foci of the quartic; so that the sixteen single foci of a bicircular quartic of the eighth class lie in fours on four confocal conics, whose common foci are the double foci of the quartic.

The conics, F, F', F'', F''', which, on account of the property just stated, I have called the focal conics of the quartic, are intimately connected with the whole theory. Thus, if F, F', &c., become circles, the quartics become Cartesian ovals; and if parabolas, the quartics reduce to circular cubics.

I have discussed Cartesian ovals from a new point of view, and have entered rather fully into their properties. Thus, being given two circles,

F and J, then, if a variable S, cutting J orthogonally, has its centre on F, its envelope is a Cartesian oval. The centre of F will be the triple focus of the oval; and the three single collinear foci will be the centre of J, and the two limiting points of J and F. I have shown, also, that the oval has six other foci, which lie two by two on three lines perpendicular to the line of collinearity of the single foci.

I have entered at some length into the properties of circular cubics. All the properties of these curves which I give in this paper I believe to be new. Thus, "being given four concyclic points," I have proved that "the two circular cubics which can be described having these points as single foci are such that the point where each intersects its asymptote is the double focus of the other;" and, again, that "the circle which has the distance between these double foci as diameter is the 'nine points' circle' of the triangle formed by any three of the four centres of inversion of either."

I have next discussed the characteristics of the various curves treated of in the paper, and of their evolutes, not only determining them for the quartics and cubics of each class, but showing the exact points and lines which are cusps, double tangents, stationary tangents, &c. and have arrived at some new theorems respecting the osculating circles of conics as well as bicircular quartics. Thus, "through any point not on an ellipse or hyperbola can be described six circles to osculate the ellipse or hyperbola, and through any point not on a bicircular quartic of the eighth class can be described twelve circles to osculate the quartic."

A very considerable portion of the paper is occupied with the application of the methods of conics to bicircular quartics. In fact, since the general equation of the second degree in α, β, γ which I employ is the same as the general equation of a conic, only that in my method the variables denote circles in place of lines, it will at once occur to any one that the methods used in the higher parts of conics apply also to bicircular quartics. I have entered very fully into this part of the subject, and have shown that the theories of invariants and covariants, reciprocation, and anharmonic ratio in conic sections, not only have their analogues in bicircular quartics, but that the very same equations and modes of proof which are employed in the one hold also in the other. In fact, this part of the paper may be regarded as an exposition of a new method of geometrical transformation; and it is shown that every graphic property of a conic section has an analogous property in bicircular quartics. Thus, "The four conics having double contact with a given conic U, which can be drawn through three given points, are all touched by four other conics having also double contact with U."

Corresponding to this we have the following theorem in bicircular quartics:—"The four bicircular quartics having quartic contact with a given bicircular quartic U, which can be described so as to have double contact with three given circles, have all double contact with four other bicircular quartics having also quartic contact with U."

I intend to follow up the mode of investigation employed in this paper in kindred parts of geometry.